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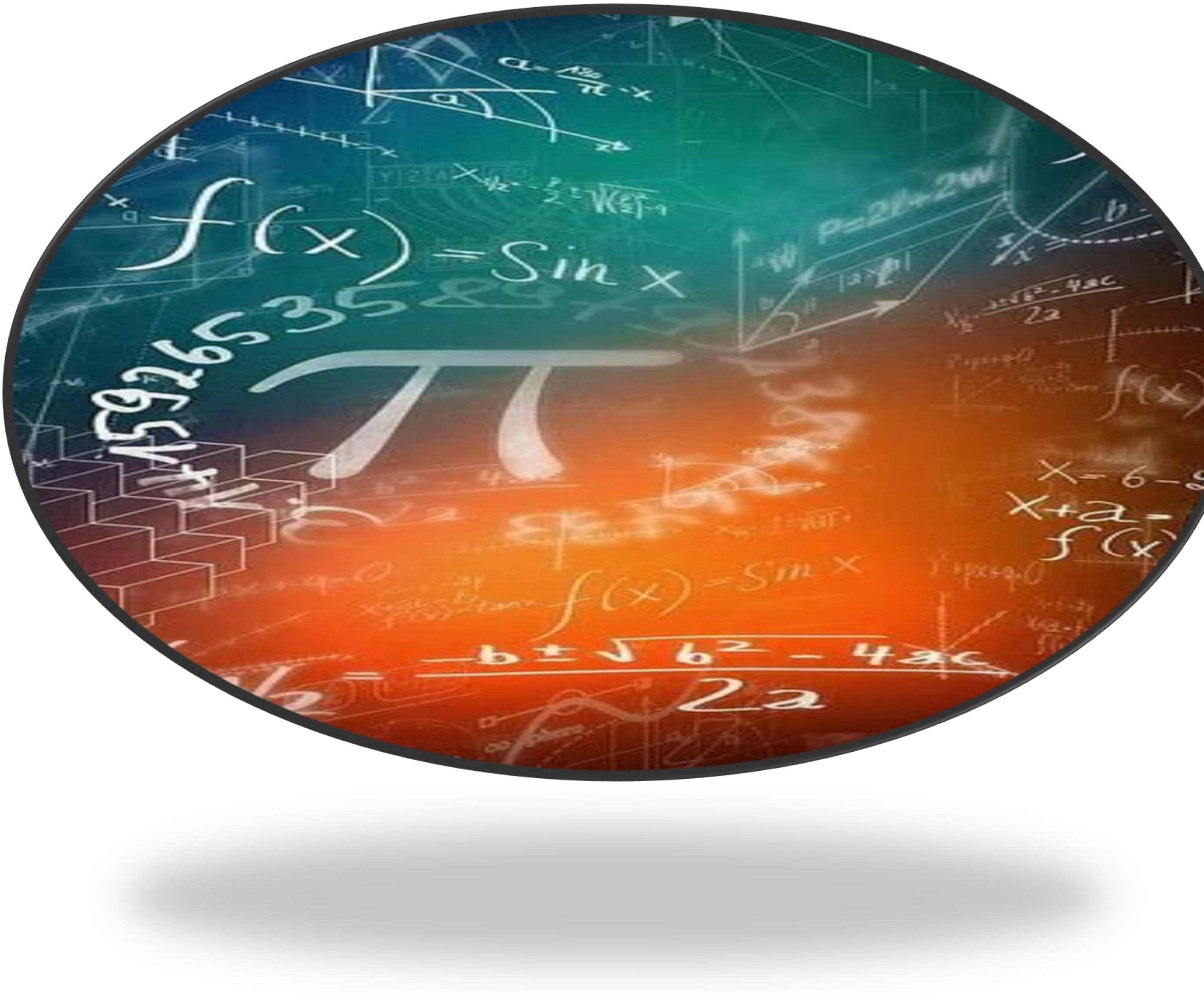
# DEPARTMENTOFBASICSCIENCE

## ENGINEERINGMATHEMATICS-I

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# CONTENT

EngineeringMathematics–I

# TRIGONOMETRY

# COMPOUNDANGLES

**INTRODUCTION :**

TheTrigonometryisderivedfromGreekwords''Trigonos''andmetrons means measurement of angles ina triangle. This subject was originallydevecpaedtosolvegeometric problems involvingtrigangles. The Hindu mathematicians Aryabhatta, Varahmira, Bramhaguptu and Bhaskar have lot of contaribution to trigonometry . Besides Hindu mathematicians ancient-Greek and Arwric mathematicians also contributed a lot to this subject. Trigonometry is used in many are as such as science of seismology, designing electrical circuits, analysing musical tones and studying the occurance of sun spots.

**TrigonometricFunctions:**

Let $q$bethemeausreofanyanglemeasuredinradians incounter clockmisesenseas showinFig(1).

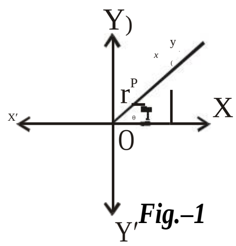
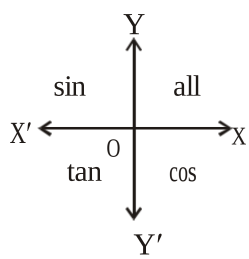


Fig.–1

LetP(x,y)beanypointantheterminalsideofangle$q$.ThedistanceofPfrom O is OP= r= $\sqrt{x^2+y^2}$ .thefunctions definedbysin$q$= $\frac{y}{r}$,cos$q$ =$\frac{x}{r}$,tan$q$=$\frac{y}{x}$ ...(1)arecalledsine, cosineandtangentfunctionsrespectirely. Thesearecalled trigonometric functions. It followrs from (1) that $\sin^2 q + \cos^2 q = 1$. Other trigonomatricfunctions such as cosecant, secant and cotangent functions are defined as cosec$q$ $=\frac{r}{y}$,sec$q$=$\frac{r}{x}$,cot$q$=$\frac{x}{y}$.

**SIGNOFT-RATIOS:**

Thestudentmayrememberthesignsoft-ratiosindifferentquadrantwiththehelpofthediagram




Thesignof paricular t-ratioinanyquadrant canberememberedbytheword''all-sin-tan-cos'' or ''add sugar to coffee''. What ever iswritten in a particular quadrant along with its reciprocal is +ve and the restare negetive.

Table giving the values of trigonometrical Ratios of angles 0º, 30°, 45º, 60º & 90º

| $\theta$ | 0º | 30º | 45º | 60º | 90º |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| $\cos\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| $\tan\theta$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | $\infty$ |

**RELATED ANGLES:**

**Definitions :** Two angles are said to be complementary angles if their sum is 90° and each angle is said to be the complement of the other.

Two angles are said to be supplementary if their sum is 180° and each angles is said to be the supplement of the other.

**To Find the T-Ratios of angle $(-\theta)$ in terms of $\theta$:**

Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle $\theta$ in the anticlockwise sense which we take as positive sense. ***(Fig. 2)***

Let OP' be the position of the radius vector after tracing $(\theta)$ in the clockwise sense, which we take as negative sense. So $\angle P'OX$ will be taken as $-\theta$. Join PP'. Let it meet OX at M.

Now $\Delta OPM \cong \Delta P'OM$, $\angle P'OM = -\theta$

$OP' = OP$, $P'M = -PM$

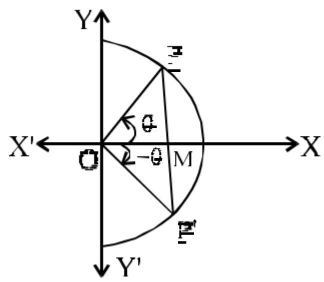


Fig.–2

$$\text{Now } \sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -\sin\theta$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -\tan\theta$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -\operatorname{cosec}\theta$$

$$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta$$

$$\cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -\cot\theta$$

**To find the T-Ratios of angle $(90°-\theta)$ in terms of $\theta$.**

Let OPM be a right angled triangle with $\angle POM = 90°$, $\angle OMP = \theta$,

$\angle OPM = 90° - \theta$. ***(Fig.3)***

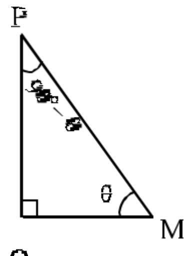


*Fig.–3*

$$\therefore \sin(90°-\theta) = \frac{OM}{PM} = \cos\theta \Rightarrow \operatorname{cosec}(90°-\theta) = \sec\theta$$

$$\cos(90°-\theta) = \frac{OP}{PM} = \sin\theta \Rightarrow \sec(90°-\theta) = \operatorname{cosec}\theta$$

$$\tan(90°-\theta)=\frac{OM}{OP}=\cot\theta \Rightarrow \cot(90°-\theta)=\tan\theta$$

**TofindtheT-Ratiosofangle(90°+θ)interms of θ.**

Let $\angle POX=\theta$ and $\angle P'OX=90°+\theta$. DrawPMandP'M'perpendicularstotheX-axis ***(Fig.4)***

Now $\Delta POM \cong \Delta P'OM'$

$\therefore$ P'M' =OMandOM' =–PM

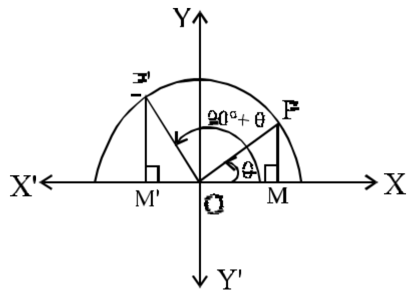



***Fig.– 4***

$$\text{Now}\sin(90°+\theta)=\frac{P'M'}{OP'}=\frac{OM}{OP}=\cos\theta$$

$$\cos(90°+\theta)=\frac{OM'}{OP'}=\frac{-PM}{OP}=-\sin\theta$$

$$\tan(90°+\theta)=\frac{P'M'}{OM'}=\frac{OM}{-PM}=-\cot\theta$$

Similarly $\operatorname{cosec}(90°+\theta)=\sec\theta$

$\sec(90°+\theta)=-\operatorname{cosec}\theta$

and $\cot(90°+\theta)=-\tan\theta$

**ToFindtheT-Ratiosofangle(180°–θ)interms ofθ.**

Let OXbetheinitialline. Let OP bethepositionoftheradius vector after tracingananangleXOP = θTo obtain theangle180° – θ let theradius vector start fromOX and after revolving through 180° cometo the position OX'. Let it revolve back through an angle θ in the clockwise direction and come to the position OP' so that the angle X'OP' is equal in magnitude but opposite in sign to the angle XOP. The angle XOP' is 180° – θ. ***(Fig.5)***

DrawP'M'andPMperpendiculartoX'OX. Now

$\Delta$ POM $\cong$ $\Delta$ P'OM'.

$\therefore$ OM'=–OMandP'M' =PM

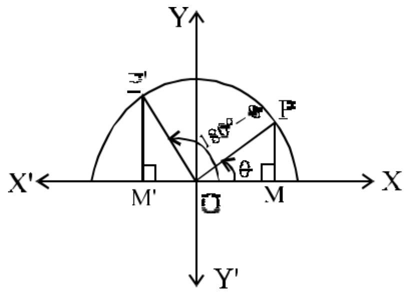



***Fig.–5***

$$\text{Now}\sin(180°-\theta)=\frac{P'M'}{OP'}=\frac{PM}{OP}=\sin\theta$$

$$\cos(180°-\theta)=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos\theta$$

$$\tan(180°-\theta)=\frac{P'M'}{OM'}=\frac{PM}{-OM}=-\tan\theta$$

Similarly $\operatorname{cosec}(180°-\theta)=\operatorname{cosec}\theta$

$\sec(180°-\theta)=-\sec\theta$

and $\cot(180°-\theta)=-\cot\theta$

**ToFindtheT-Ratiosofangle(180°+θ)interms ofθ.**

Let $\angle POX=\theta$ and $\angle P'OX=90°+\theta$. ***(Fig.6)***

Now $\Delta POM \cong \Delta P'OM'$.

$\therefore$ OM'=–OMandP'M' =–PM

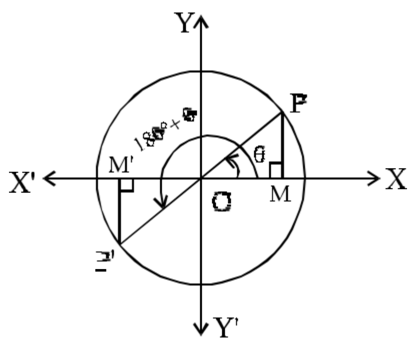



***Fig.–6***

$$\text{Now}\sin(180°+\theta)=\frac{P'M'}{OP'}=\frac{-PM}{OP}=-\sin\theta$$

$$\cos(180°+\theta)=\frac{OM'}{OP'}=\frac{-OM}{OP}=-\cos\theta$$

$$\tan(180°+\theta)=\frac{P'M'}{OM'}=\frac{-PM}{-OM}=\tan\theta$$

Similarly $\operatorname{cosec}(180°+\theta)=\operatorname{cosec}\theta$

$\sec(180°+\theta)= -\sec\theta$

and $\cot(180°+\theta)=\cot\theta$.

**To Find the T-Ratios of angles $(270°\pm\theta)$ in terms of $\theta$.**

The trigonometrical ratios of $270° - \theta$ and $270° + \theta$ in terms of those of $\theta$, can be deduced from the above articles. For example

$\sin(270°-\theta) = \sin[180° + (90°-\theta)]$

$= -\sin(90°-\theta) = -\cos\theta$

$\cos(270° -\theta) = \cos[180°+(90°-\theta)]$

$= -\cos(90°-\theta) = -\sin\theta$

Similarly $\sin(270°+\theta) = \sin[180°+(90°+\theta)]$

$= -\sin(90°+\theta) = -\cos\theta$

$\cos(270°+\theta) = \cos[180°+(90°+\theta)]$

$= -\cos(90°+\theta) = -(-\sin\theta) = \sin\theta$

**To Find the T-Ratios of angles $(360°\pm\theta)$ in terms of $\theta$.**

We have seen that if n is any integer, the angle $n.\,360°\pm\theta$ is represented by the same position of the radius vector as the angle $\pm\theta$. Hence the trigonometrical ratios of $360°\pm\theta$ are the same as those of $\pm\theta$. Thus $\sin(n.360°+\theta) = \sin\theta$

$\cos(n.360°+\theta) = \cos\theta$

$\sin(n.360° -\theta) = \sin(-\theta) = -\sin\theta$

and $\cos(n.360°-\theta) = \cos(-\theta) = \cos\theta$.

***Examples:***

$\cos(-720°-\theta) = \cos(-2\times 360°-\theta) = \cos(-\theta) = \cos\theta$ and $\tan(1440° +\theta) = \tan(4\times 360° +\theta) = \tan\theta$

In general when is any integer, $n \in Z$

(1) $\sin(n\pi+\theta) = (-1)^n \sin\theta$

(2) $\cos(n\pi+\theta) = (-1)^n \cos\theta$

(3) $\tan(n\pi+\theta) = \tan\theta$ when n is odd integer

(4) $\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}} \cos\theta$

(5) $\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}} \sin\theta$

(6) $\tan\left(\frac{n\pi}{2}+\theta\right) = \cot\theta$

**EVEN FUNCTION:**

A function $f(x)$ is said to be an even function of $x$, if $f(x)$ satisfies the relation $f(-x) = f(x)$. **Ex.** cosx, secx, and all even powers of x i.e, $x^2, x^4, x^6 \ldots\ldots$ are even function.

**ODD FUNCTION:**

A function $f(x)$ is said to be an odd function of $x$, if $f(x)$ satisfies the relation $f(-x) = -f(x)$.

**Ex.** sinx, cosecx, tanx, cotx and all odd powers of x i.e, $x^3$, $x^5$, $x^7 \ldots\ldots$ are odd function.

**Example:** ***Find the values of $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{-12}{13}$ and $\theta$ lies in the third quadrant.***

***Solution :*** We have $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta}$$

In third quadrant $\sin\theta$ is negetive, therefore

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = \frac{-5}{3}$$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-5}{13} \times \frac{13}{-12} = \frac{5}{12}$

**Example: Find the values of**

**(i) $\tan(-900^\circ)$ (ii) $\sin 1230^\circ$**

***Solution:*** (i) $\tan(-900^\circ) = -\tan 900^\circ = -\tan(10 \times 90^\circ + 0^\circ) = -\tan 0^\circ = 0$

(ii) $\sin(1230^\circ) = \sin(6\times180^\circ+150^\circ) = \sin 150^\circ = \sin(180^\circ-30^\circ) = \sin 30^\circ = \frac{1}{2}$

**Example: Show that**

$$\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta).\cot(90^\circ-\theta)} = -1$$

***Solution:***

$$\frac{\cos(90^\circ+\theta).\sec(-\theta).\tan(180^\circ-\theta)}{\sec(360^\circ-\theta).\sin(180^\circ+\theta).\cot(90^\circ-\theta)} = \frac{-\sin\theta \times \sec\theta \times -\tan\theta}{\sec\theta \times -\sin\theta \times \tan\theta} = -1$$

## ASSIGNMENT

1. Find the value of $\cos 1^\circ.\cos 2^\circ \ldots\ldots \cos 100^\circ$
2. Evaluale: $\tan\frac{\pi}{20} \times \tan\frac{3\pi}{20} \times \tan\frac{5\pi}{20} \times \tan\frac{7\pi}{20} \times \tan\frac{9\pi}{20}$

## COMPOUND,MULTIPLEANDSUB-MULTIPLEANGLES

Whenanangleformedasthealgebricsumoftwoor moreanglesiscalleda compoundangles. Thus A + B and A + B + c are compound angles.

### *AdditionFormulae*

Whenanangleformedas thealgebraicalsumoftwoor moreangles, it iscalleda compoundangles. Thus A + B and A + B + C are compound angles.

**AdditionFormula:**

(i) sin(A+ B)=sin A.cos B+ cos A.sinB

(ii) cos(A+ B)=cos A.cos B – sinA.sinB

(iii) $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A.\tan B}$

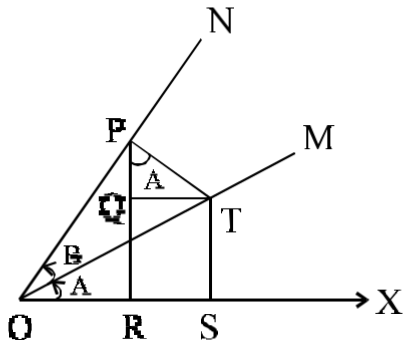



*Fig. –7*

***Proof:*** Let therevolvinglineOM startingfromthelineOXmake an angle XOM = A and then further move to make.

ÐMON=B,sothatÐXON=A +B*(Fig.7)*

Let'P'beanypointonthelineON.

DrawPR^OX,PT^OM,TQ^PRandTS^OX

ThenÐQPT =90°–ÐPTQ=ÐQTO=ÐXOM=A

$\therefore$ Wehavefrom ÐOPR

(i) $\sin(A+B) = \dfrac{RP}{OP} = \dfrac{QR+PQ}{OP} = \dfrac{TS+PQ}{OP}$ (QQR=TS)

$= \dfrac{TS}{OP}+\dfrac{PQ}{OP} = \dfrac{TS}{OT}.\dfrac{OT}{OP}+\dfrac{PQ}{PT}.\dfrac{PT}{OP}$

$=\sin A.\cos B+\cos A.\sin B$

(ii) $\cos(A+B) = \dfrac{OR}{OP} = \dfrac{OS-RS}{OP} = \dfrac{OS}{OP}-\dfrac{RS}{OP}$

$= \dfrac{OS}{OP}-\dfrac{QT}{OP}$ [QRS=QT]

$= \dfrac{OS}{OT}.\dfrac{OT}{OP}-\dfrac{QT}{PT}.\dfrac{PT}{OP}$

$=\cos A.\cos B-\sin A.\sin B$

(iii) $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B)}$

(dividingnumerator anddenominator bycosAcosB)

$= \dfrac{\dfrac{\sin A\cos B}{\cos A\cos B}+\dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B}-\dfrac{\sin A\sin B}{\cos A\cos B}}$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A.\tan B}$$

(iv) $\cot(A+B) = \dfrac{\cos(A+B)}{\sin(A+B)}$

$= \dfrac{\cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B}$

[dividingofnumeratoranddenominatorbysinAsinB]

$= \dfrac{\dfrac{\cos A\cos B}{\sin A\sin B} - 1}{\dfrac{\sin A\cos B}{\sin A\sin B} + \dfrac{\cos A\sin B}{\sin A\sin B}}$

$\cot(A+B) = \dfrac{\cot A.\cot B - 1}{\cot B + \cot A}$

***Cor:*** In theaboveformulae,replacingA by $\frac{\pi}{2}$ and Bby x

Wehave

(i) $\sin\left(\frac{\pi}{2}+x\right) = \sin\frac{\pi}{2}.\cos x + \cos\frac{\pi}{2}.\sin x$

$= 1.\cos x + 0.\sin x = \cos x$

(ii) $\cos\left(\frac{\pi}{2}+x\right) = \cos\frac{\pi}{2}.\cos x - \sin\frac{\pi}{2}.\sin x$

$= 0\times\cos x - 1\times\sin x = -\sin x$

(iii) $\tan\left(\frac{\pi}{2}+x\right) = \dfrac{\sin\left(\frac{\pi}{2}+x\right)}{\cos\left(\frac{\pi}{2}+x\right)} = \dfrac{\cos x}{-\sin x} = -\cot x$

**(b) DifferenceFormulae:**

(i) $\sin(A-B) = \sin A.\cos B - \cos A.\sin B$

(ii) $\cos(A-B) = \cos A.\cos B + \sin A.\sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A.\tan B}$

***Proof :*** LetthereveolvinglineOMmakeanangleAwithOXandthenresolvebacktomake $\angle MON = B$ so that $\angle XON = A - B$. ***(Fig. 8)***

Let 'P' be any point on ON. Draw $PR \perp OX$,

$PT \perp OM$, $TS \perp OX$, $TQ \perp RP$ producedtoQ.

Then $\angle TPQ = 90° - \angle PTQ = \angle QTM = A$

Nowfrom $\Delta OPR$,wehave

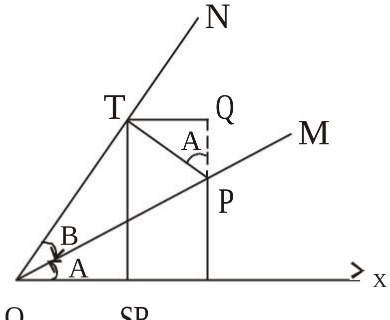



***Fig. –8***

(i) $\sin(A-B) = \dfrac{PR}{OP} = \dfrac{QR - QP}{OP} = \dfrac{TS - QP}{OP}$

$= \frac{TS}{OP} \cdot \frac{QP}{OP}$

$= \frac{TS}{OT} \cdot \frac{OT}{OP} - \frac{OP}{PT} \cdot \frac{PT}{OP}$

$= \sin A . \cos B - \cos A . \sin B$

(ii) $\cos(A-B) = \frac{OR}{OP} = \frac{OS+SR}{OP} = \frac{OS+TQ}{OP} = \frac{OS}{OP} + \frac{TQ}{QP}$

$= \frac{OS}{OT} . \frac{OT}{OP} + \frac{TQ}{PT} . \frac{PT}{OP}$

$= \cos A . \cos B + \sin A . \sin B$

(iii) $\tan (A - B) = \frac{\sin(A-B)}{\cos (A - B)} = \frac{\sin A . \cos B - \cos A . \sin B}{\cos A \cos B + \sin A \sin B}$

$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Dividingthenumeratorandthedenominatorbycos A.cosB.

(iv) $\cot(A-B) = \frac{\cos ( A - B)}{\sin(A-B)}$

$= \frac{\cos A . \cos B + \sin A . \sin B}{\sin A . \cos B - \cos A . \sin B}$

$= \frac{\cot A . \cot B + 1}{\cot B - \cot A}$

dividingthenumeratoranddenominator bysinA. sinB

Wecanalsodeducesubstractionformulaefromadditionformulae inthefollowing manner. $\sin(A - B) = \sin[A + (-B)]$

$= \sin A . \cos (-B) + \cos A . \sin(-B)$

$= \sin A . \cos B + \cos A .$

$\sin B$ $\cos(A-B) = \cos[A+(-B)]$

$= \cos A . \cos (-B) - \sin A . \sin(-B)$

$= \cos A . \cos B + \sin A . \sin B$

$\tan(A-B) = \tan[A+(-B)] = \frac{\tan A + \tan(-B)}{1 - \tan A . \tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A . \tan B}$

***Example–1 :*** **Findthevalueoftan75°andhenceprovethattan75°+cot75°= 4**

***Solution:*** $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1 \times 1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$\therefore \tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

Engineering Mathematics – I

$\therefore \cot 75° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ $\left(\text{since } \cot\theta = \dfrac{1}{\tan\theta}\right)$

$\tan 75° + \cot 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} + \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \dfrac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$

$= \dfrac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1}$ [since $(a+b)(a-b)=a^2-b^2$]

$\therefore \tan 75° + \cot 75° = 4$

***Example–2:* If $\sin A = \dfrac{1}{\sqrt{10}}$ and $\sin B = \dfrac{1}{\sqrt{5}}$ show that $A + B = \dfrac{\pi}{4}$**

***Solution:*** $\sin A = \dfrac{1}{\sqrt{10}}$

$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\dfrac{1}{10}} = \sqrt{\dfrac{10-1}{10}} = \sqrt{\dfrac{9}{10}}$

$\therefore \cos A = \dfrac{3}{\sqrt{10}}$

$\sin B = \dfrac{1}{\sqrt{5}}, \cos B = \sqrt{1-\sin^2 B}$

$= \sqrt{1-\dfrac{1}{5}} = \sqrt{\dfrac{5-1}{5}} = \sqrt{\dfrac{4}{5}}$

$\therefore \cos B = \dfrac{2}{\sqrt{5}}$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \dfrac{1}{\sqrt{10}} \times \dfrac{2}{\sqrt{5}} + \dfrac{3}{\sqrt{10}} \times \dfrac{1}{\sqrt{5}} = \dfrac{2}{\sqrt{50}} + \dfrac{3}{\sqrt{50}}$

$= \dfrac{2+3}{\sqrt{50}} = \dfrac{2+3}{5\sqrt{2}}$

$\therefore \sin(A+B) = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\sin(A+B) = \sin 45°$

$\therefore A+B = 45° = \dfrac{\pi}{4}$ $\left(\text{since } 45° = \dfrac{180°}{4}\right)$

***Transformation of Sums or Differences into Products***

(a) We have that

$\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ .................... (1)

$\sin(A+B)-\sin(A-B) = 2\cos A\sin B$ ....................(2)

$\cos(A+B)-\cos(A-B) = 2\cos A\cos B$..................(3)

$\cos(A-B)-\cos(A+B) = 2\sin A\sin B$....................(4)

Let $A+B=C$ and $A-B=D$

Then $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

Puttingthevalueinformula(1),(2),(3)and(4)weget

$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ ............................(i)

$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$ ............................(ii)

$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$ ............................(iii)

$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$ ............................(iv)

forpracticeitismoreconvenienttoquotetheformulaeverballyasfollows:
Sum of two sines = 2 sin (half sum) cos (half difference)
Differenceoftwosines=2cos(halfsum)sin(halfdifference) Sum of two cosines = 2 cos (half sum) cos (half difference)
Differenceoftwocosines=2sin(halfsum)sin(halfdifferencereversed)

[Thestudentshouldcarefullynoticethatthesecondfactoroftherighthandmemberof IVissin $\frac{D-C}{2}$, notsin $\frac{C-D}{2}$ ]

**(b) To findtheTrigonometricalratios ofAngle2Ainterms ofthoseofA:sin2A, cos 2A.**

Since $\sin(A+B)=\sin A\cos B+\cos A\sin B$

putting$B=A$

$\sin(A+A)=\sin A\cos A+\cos A\sin A$

$\Rightarrow \sin 2A=2\sin A\cos A$ .................................. (i)

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\Rightarrow \cos(A+A)=\cos A\cos A-\sin A\sin A$

$\Rightarrow \cos 2A=\cos^2A-\sin^2A$ .....................................(ii)

Also$\cos 2A=1-\sin^2A-\sin^2A=1-2\sin^2A$ ............................ (iii)

So $2\sin^2A=1-\cos 2A$ .....................................................(iv)

Also$\cos 2A=\cos^2A-(1-\cos^2A)=2\cos^2A-1$ .......................(v)

or $2\cos^2A=1+\cos 2A$ .................................................... (vi)

**(c) Formulafortan2A**

sincetan $(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$\tan 2A=\tan(A+A)=\frac{\tan A+\tan A}{1-\tan A\tan A}$

$=\frac{2\tan A}{1-\tan^2A}$

Notethisformulaisnotdefinedwhen$\tan^2A=1$ i.e,$\tan A=\pm 1$

**(d) Toexpress sin2Aandcos2Aintermsoftan A**

$\sin 2A=2\sin A\cos A$

$=2\frac{\frac{\sin A}{\cos A}}{\frac{1}{\cos^2A}}=\frac{2\tan A}{\sec^2A}=\frac{2\tan A}{1-\tan^2A}$

Also, $\cos 2A = \cos^2 A - \sin^2 A$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

(dividing numerator and denominator by $\cos^2 A$)

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

**(e) To find the Trigonometrical formulae of 3A**

$\sin 3A = \sin(2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A$

$= 2\sin A \cos A . \cos A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A$

$= 3\sin A - 4\sin^3 A$

Again, $\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin A \cos A . \sin A$

$= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A)$

$= 4\cos^3 A - 3\cos A$

Also $\tan 3A = \tan(2A + A)$

$$= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2\tan A}{1 - \tan^2 A} . \tan A}$$

$$= \frac{2\tan A + \tan A(1 - \tan^2 A)}{1 - \tan^2 A - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}, \text{ provided } 3\tan^2 A \neq 1 \text{ i.e, } \tan A \neq \pm \frac{1}{\sqrt{3}}$$

**(f) Submultiple Angles:**

To express trigonometric ratios of A in terms of ratios of A/2

$\sin 2\theta = 2\sin\theta\cos\theta$ (true for all value of $\theta$)

Let $2\theta = A$ i.e. $\theta = \frac{A}{2}$

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ .... (i)

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

or $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$ ..... (ii)

$\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$ .....(iii)

**Also, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$**

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \quad \text{........................(iv)}$$

[Where $A \neq n\pi \frac{\pi}{2}$, $(n\in I)$ and $A \neq (2n+1)\pi$]

Again, $\sin A = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

[dividing numerator and denomenator by $\cos^2\frac{A}{2}$)

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

[where $A \neq (2n+1)\pi, n\in I$]

Similarly, $\cos A = \frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{1} = \frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}}$

Now dividing numerator and denominator by $\cos^2\frac{A}{2}$

$$\Rightarrow \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} \text{ [where } A \neq (2n+1)\pi, n\in I].$$

**Example–1: Find the values of**

**(i) $\cos 22\frac{1}{2}^\circ$ (ii) $\sin 15^\circ$**

***Solution:*** **(i)** We have $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$, putting $A = 45^\circ$

$$\cos 22\frac{1}{2}^\circ = \sqrt{\frac{1+\cos 45^\circ}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

**(ii)** $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ.\cos 30^\circ - \cos 45^\circ.\sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

**Example – 2: Prove that $\sin A.\sin(60^\circ - A).\sin(60^\circ + A) = \frac{1}{4}\sin 3A$**

***Solution:*** $\sin A.\sin(60^\circ - A)\sin(60^\circ + A)$

$$= \sin A.(\sin^2 60^\circ - \sin^2 A) \qquad [\because \sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B]$$

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right] = \sin A\left[\frac{3}{4} - \sin^2 A\right] = \frac{1}{4}[3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A$$

**Example – 3: Prove that $\sin 20^\circ.\sin 40^\circ.\sin 60^\circ.\sin 80^\circ = \frac{3}{16}$**

***Solution:*** $\sin 60^\circ.\sin 20^\circ.\sin 40^\circ.\sin 80^\circ$

$$= \frac{\sqrt{3}}{2}[\sin A.\sin(60^\circ - A).\sin(60^\circ + A)] \text{ where } A = 20^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{4} \times \sin 3A = \frac{\sqrt{3}}{8} \times \sin 60 = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2} = \frac{3}{16}$$

**Example – 4: If $A + B + C = \pi$ and $\cos A = \cos B.\cos C$ show that $\tan B + \tan C = \tan A$**

***Solution:*** L.H.S. $= \tan B + \tan C$

$$= \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin B.\cos C + \cos B.\sin C}{\cos B.\cos C}$$

$$= \frac{\sin(B+C)}{\cos B.\cos C} = \frac{\sin(\pi - A)}{\cos B.\cos C} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S. (Proved)}$$

**Examples – 5: Prove the followings**

(a) **$\cot 7\frac{1^\circ}{2} = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$**

(b) **$\tan 37\frac{1^\circ}{2} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$**

***Solution:* (a)** We know $\cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$ (Choosing $\theta = 15$)

$$\Rightarrow \cot 7\frac{1^\circ}{2} = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1 + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{(2\sqrt{2}+\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+1+\sqrt{3}}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{2} = \sqrt{6}+\sqrt{3}+\sqrt{2}+2$$

**(b)** We know $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ (Choosing $\theta = 75°$)

$$\tan 37\tfrac{1}{2}° = \frac{1-\cos 75°}{\sin 75°} = \frac{1-\cos(90°-15°)}{\sin(90°-15°)}$$

$$= \frac{1-\sin 15°}{\cos 15°} = \frac{1-\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}+1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

**Example– 6: If $\sin A = K\sin B$, prove that $\tan\frac{1}{2}(A-B) = \frac{K-1}{K+1}\tan\frac{1}{2}(A+B)$**

***Solution:*** Given $\sin A = K\sin B$

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{K}{1}$$

Using componendo & dividendo

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{K+1}{K-1}$$

$$\Rightarrow \frac{2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}.\sin\frac{A-B}{2}} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2}.\cot\frac{A-B}{2} = \frac{K+1}{K-1}$$

$$\Rightarrow \tan\frac{A+B}{2} = \frac{K+1}{K-1}.\tan\frac{A-B}{2}$$

$$\Rightarrow \tan\frac{A-B}{2} = \frac{K-1}{K+1}\tan\frac{A+B}{2}$$

$\therefore$ L.H.S. = R.H.S. (Proved)

**Example–7: If $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$, Prove that $\cos\beta = \frac{\cos\alpha - e}{1-e\cos\alpha}$**

***Solution:*** $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$ (Given)

$$\tan^2\frac{\beta}{2} = \frac{1+e}{1-e}\tan^2\frac{\alpha}{2}$$

L.H.S $= \cos\beta$

$$= \frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}} = \frac{1-\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}{1+\frac{1+e}{1-e}\tan^2\frac{\alpha}{2}}$$

$$= \frac{1-e-\tan^2\frac{\alpha}{2}-e\tan^2\frac{\alpha}{2}}{1-e+\tan^2\frac{\alpha}{2}+e\tan^2\frac{\alpha}{2}} = \frac{\left(1-\tan^2\frac{\alpha}{2}\right)-e\left(1+\tan^2\frac{\alpha}{2}\right)}{\left(1+\tan^2\frac{\alpha}{2}\right)-e\left(1-\tan^2\frac{\alpha}{2}\right)}$$

$$= \frac{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} - e\dfrac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{\dfrac{1+\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} - e\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}$$

$$= \frac{\cos\alpha - e}{1-e\cos\alpha} = \text{R.H.S(Proved)}$$

**Example– 8: If $A+B+C=\pi$, then Prove the following**

**(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A.\sin B.\sin C$**

***Solution:*** L.H.S.$=\sin 2A+\sin 2B+\sin 2C$

$=2\sin(A+B).\cos(A-B)+2\sin C.\cos C$

$=2\sin(\pi-C).\cos(A-B)+2\sin C.\cos C$

$=2\sin C.\cos(A-B)+2\sin C.\cos C$

$=2\sin C[\cos(A-B)+\cos C]$

$=2\sin C[\cos(A-B)-\cos(A+B)]$

$=2\sin C.2\sin A.\sin B$

$=4\sin A.\sin B.\sin C$ R.H.S.(Proved)

**(ii) $\sin 2A+\sin 2B-\sin 2C=4\cos A.\cos B.\sin C$**

***Solution:*** L.H.S.$=\sin 2A+\sin 2B-\sin 2C$

$=2\sin(A+B).\cos(A-B)-2\sin C.\cos C$

$=2\sin(\pi-C).\cos(A-B)-2\sin C.\cos C$

$=2\sin C.\cos(A-B)-2\sin C.\cos C$

$=2\sin C[\cos(A-B)-\cos\{\pi-(A+B)\}]$

$=2\sin C\{\cos(A-B)+\cos(A+B)\}$

$$=2\sin C\left\{2\cos\frac{A-B+A+B}{2}.\cos\frac{A-B-A-B}{2}\right\}$$

$=4\sin C.\cos A.\cos B.$

**(iii) $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$**

***Solution:*** L.H.S.$=\sin A+\sin B-\sin C$

$$=2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}-2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}.\cos\frac{A-B}{2}-2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[-2\sin\left(\frac{\frac{A-B}{2}+\frac{A+B}{2}}{2}\right)\sin\left(\frac{\frac{A-B}{2}-\frac{A+B}{2}}{2}\right)\right]$$

$$= -4\cos\frac{C}{2}.\sin\frac{A}{2}.\sin\left(-\frac{B}{2}\right)$$

$$= 4\cos\frac{C}{2}.\sin\frac{A}{2}.\sin\frac{B}{2} = \text{R.H.S (Proved)}$$

## ASSIGNMENT

**1.** If $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, then find the value of $(\alpha + \beta)$

**2.** Find the value of $\dfrac{\cos 15^\circ + \sin 15^\circ}{\cos 15^\circ - \sin 15^\circ}$

**3.** Prove that $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

**4.** If $A + B = 45^\circ$, show that $(1 + \tan A)(1 + \tan B) = 2$

**5.** If $(1-e)\tan^2\frac{\beta}{2} = (1+e)\tan^2\frac{\alpha}{2}$

Prove that $\cos\beta = \dfrac{\cos\alpha - e}{1 - e\cos\alpha}$

**6.** If $A+B+C=\pi$, prove that

$\cos 2A + \cos 2B + \cos 2C + 1 + 4\cos A.\cos B.\cos C = 0$

# FUNCTION AND LIMIT

Letsdiscusswhatafunction is

Afunctionisbasicallyarulewhichassociatesanelementwithanother element.

Therearedifferentrulesthatgoverndifferentphenomenaorhappeningsin our day to day life.

For example,

i. Waterflowsfromahigheraltitudetoalower altitude
ii. Heatflowsfromhighertemperaturetoalowertemperature.
iii. Externalforceresultsinchangestateofabody(Newton's1$^{st}$Ruleof motion) etc.

Alltheserulesassociatesaneventorelementtoanothereventorelement, say , x with y.

Mathematicallywewrite,

$$y=f(x)$$

i.e.giventhevalueofxwecandeterminethevalueofybyapplyingtherule 'f'

for example,

$$y=x+1$$

i.ewecalculatethevalueofybyadding1tovalueofx.Thisistheruleor function we are discussing.

Since we say a function associates two elements, x and y we can think of twosetsAandBsuchthatxistakenfromsetAandyistakenfromsetB. Symbolicallywewrite

$x\varepsilon$ A(xbelongsto A)

$y \varepsilon B$ (xbelongs toB)

$y = f(x)$ canalsobewrittenas

$(x,y)\ \varepsilon\ f$

Since(x,y)representsapair ofelementswecanthinkoftheseinrelations $f \subseteq A \times B$ or

fcanthoughtofasasubsetoftheproductofsetsAandBwe haveearlier referred to.

And,therefore,theelementsoffarepairofelementslike(x,y).

Inthediscussionofafunctionwemustconsiderall theelementsofsetA and see that no x is associated with two different values of y in the set B

Whatisdomainof function

Since function associates elements x of A to elements y of B and function must take care of all the elements of set A we call the set A as domain of thefunction.Wemusttakenoteofthefactthatifthefunctioncannotbe defined for some elements of set A , the domain of the function willbe a subset of A.

Example1

Let $A=\{1,2,3,4,-1,0,-4\}$

$B=\{0,1,2,3,4,-1,-2,-3\}$

Thefunctionisgivenby $y = f(x) = x + 1$

forx=1,y=2

x=2, y=3

x=3,y=4

x=4,y=5

x=-1,y=0

x=0,y=1

x=-4,y=-3

clearlyy=5andy=-3 donotbelongtosetB.thereforewesaythedomainof this function is

theset$\{0,1,2,3,-1,\}$whichisasubsetofsetA. What is

range of a function

Range of the function is the set of all y’s whose values are calculated by taking all the values of x in the domain of the function. Since the domain of thefunctioniseither isequaltoAorsubsetofsetA,rangeofthefunction is either equal to set b or sub set of setB.

Intheearlier example,

Rangeoffunctionistheset $\{1,2,3,4,0\}$whichisasubsetofsetB

SOMEFUNDAMENTAL FUNCTIONS

*Constant Function*

Y=f(x)=K,forallx

Therulehereis:thevalueofyisalwaysk,irrespectiveofthevalueofx

Thisisaverysimpleruleinthesensethat evaluationofthevalueofyisnot required as it is already given as k

Domainof‘f’issetofallrealnumbers

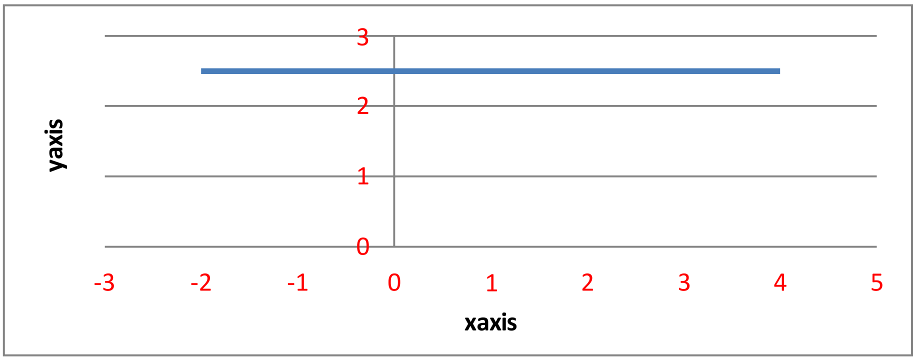
Rangeof‘f’isthesingletonsetcontaining‘k’alone. Or

Dom=R,setofallrealnumbers Range=

{k}

GraphofConstantFunction

Lety=f(x)=k =2.5




Thegraphisalineparalleltoaxisofx

## *Identity Function*

Y=f(x)=x,forallx

Therulehereis:thevalueofyisalways equalstox

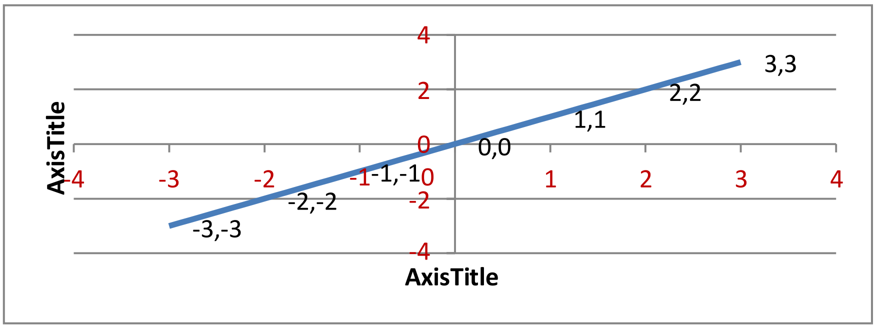
Thisisalsoaverysimpleruleinthesensethatthevalueofyisidentical with the value of x saving our time to calculate the value of y.

Dom = R

Range=R

i.e.DomainofthefunctionissameasRangeofthefunction Graph of

Identity Function



*ModulusFunction*

$$y=f(x)=|x|=\begin{cases} x, x\geq 0 \\ -x, x<0 \end{cases}$$

Theruleherereis:thevalueofyisalwaysequalstothenumericalvalueofx, not taking in to consideration the sign of x.

Example

Y=f(2)=2

Y=f(0)=0

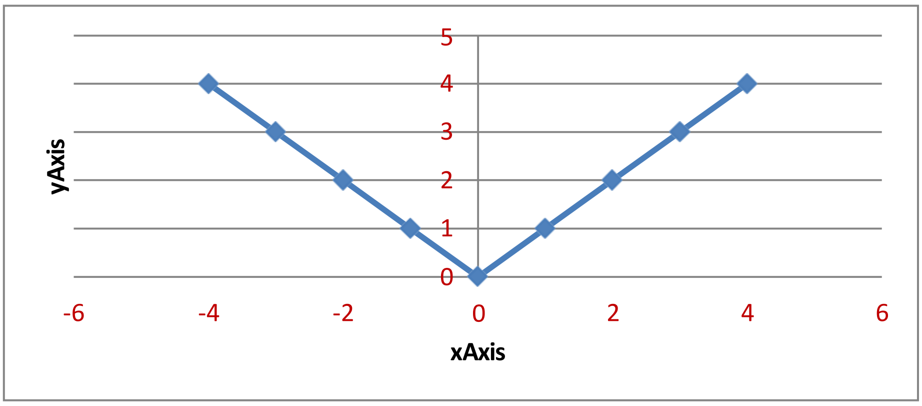
Y=f(-3)=3

Thisfunctionisusuallyusefulindealingwithvalueswhicharealways positive for example, length, area etc.

Dom=R

Range=$R^{+}$U{0}

GraphofModulusFunction




*SignumFunction*

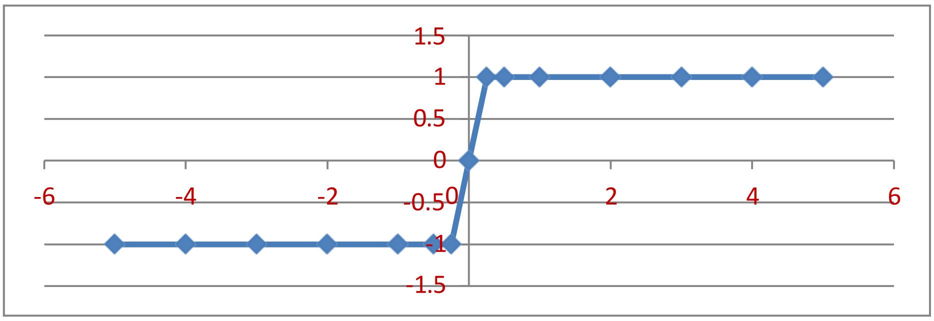
$$y=f(x)=\begin{cases} \frac{|x|}{x}, x\neq 0 \\ 0, x=0 \end{cases}=\begin{cases} 1, x>0 \\ 0, x=0 \\ -1, x<0 \end{cases}$$

Thisisalsoaverysimpleruleinthesensethatthevalueofyis 1ifxis positive , 0 when x=0, and -1 when x is negative.

Dom=R

Range=$\{-1,0,1\}$

GraphofSignumFunction




*GreatestIntegerFunction*

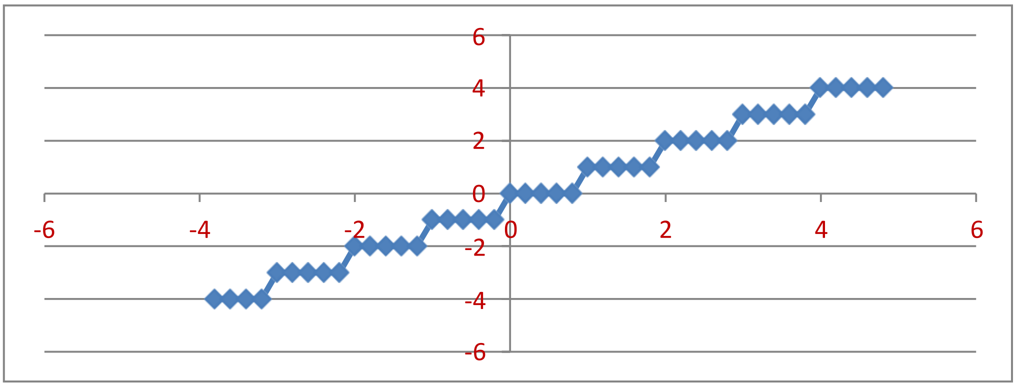
$$y=f(x)=[x]=greatestinteger\leq x$$

For Example $[0]=0,[0.2]=0,[2.5]=2,[-3.8]=-4,etc.$

Dom=R

Range=Z(setofallIntegers)

Graph of The function




*ExponentialFunction*

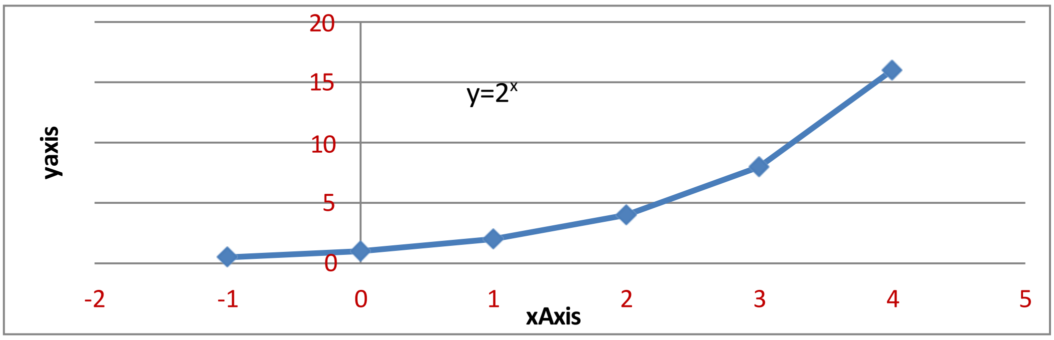
$$y=f(x)=a^{x}wherea>0$$

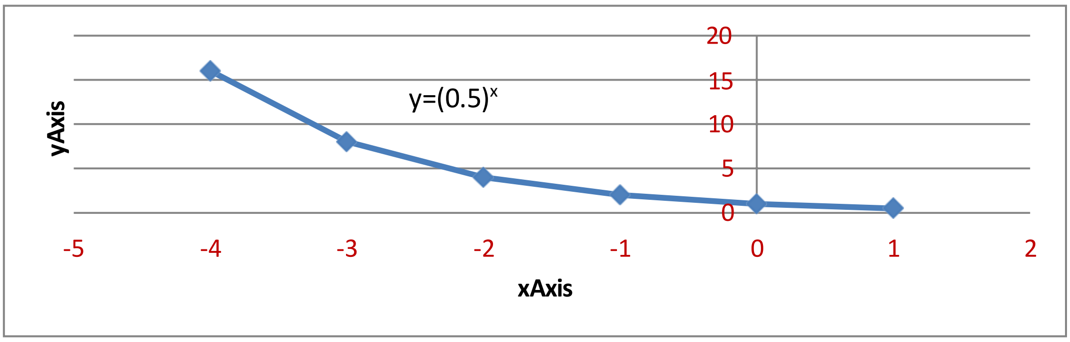
Dom=R

Range=R+

Thespecialtyofthefunctionisthatwhateverthevalueofx,ycanneverbe 0 or negative

GraphofExponentialFunction



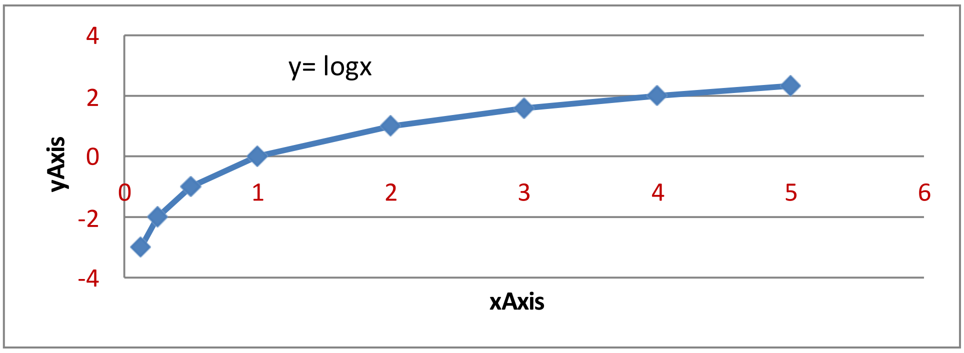

*LogarithmicFunctio n*

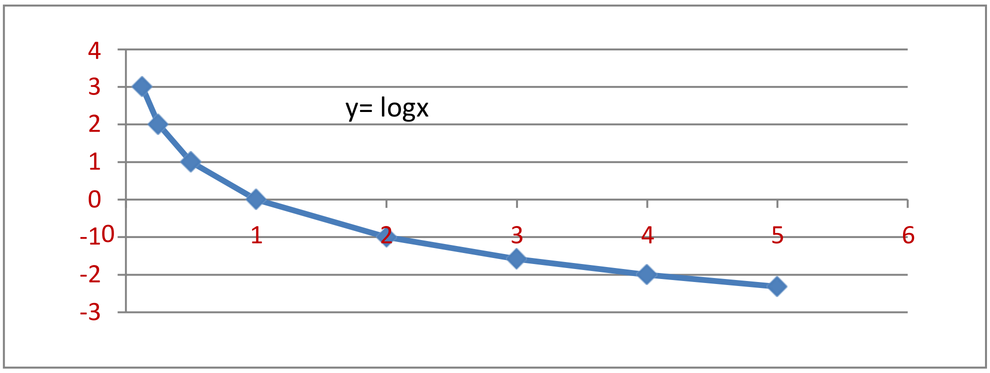
$$y=f(x)=\log_a x$$

Dom=R+ Range

=

GraphofLogarthmicFunction




y= logx
4
3
2
1
0
-10
-2
-3
1
2
3
4
5
6

LIMITOFA FUNCTION

Considerthefunction y

=2 x + 1

letsseewhathappenstovalueofyasthevalueofx changes.

Lets take the values of x close to the value of, say, 2. Now when we say valueof x close2. Itcanbe a valuelike2.1or 1.9. inonecase itis close to 2 butgreaterthan2andinotheritis closeto2butlessthan2.Nowconsider a sequence of such numbers slightly greater than 2 and slightly less than 2 and accordingly calculate the value of y in each case.

Lookatthetable

| x | y=2x+1 |
|---|---|
| 1.9 | 4.8 |
| 1.91 | 4.82 |
| 1.92 | 4.84 |
| 1.93 | 4.86 |
| 1.94 | 4.88 |
| 1.95 | 4.9 |
| 1.96 | 4.92 |
| 1.97 | 4.94 |
| 1.98 | 4.96 |
| 1.99 | 4.98 |
| 2.01 | 5.02 |
| 2.02 | 5.04 |
| 2.03 | 5.06 |
| 2.04 | 5.08 |
| 2.05 | 5.1 |
| 2.06 | 5.12 |
| 2.07 | 5.14 |
| 2.08 | 5.16 |
| 2.09 | 5.18 |
| 2.1 | 5.2 |

Weseeinthetabulatedvalue that

asxisapproachingthevalueof2fromeitherside,thevalueofyis approaching the value of5

inotherwordswesay,

y→5(ytendsto 5)asx→2(xtendsto2)or

$$\lim_{x\to 2} \quad y=5$$

INFINITELIMIT

Asx→aforsomefinitevalueofa,ifthevalueofy isgreaterthananypositive numberhowever large then we say

Y→∞(ytendstoinfinity)

Inotherwordsyissaidhaveaninfinitelimitasx→a.Andwe write

$$\lim_{x\to a} y=\infty$$

Example

If

$y=\frac{1}{x^2}$,

Then

$$\lim_{x\to 0} y=\infty$$

Sincex→0, $x^2\to 0 and x^2$ispositive,

$\frac{1}{x^2}$becomes veryvery largeandispositive.Thereforetheresult.

Similarly,

Asx→aforsomefinitevalueofa,ifthevalueofyislessthananynegative numberhowever large then we say

Y→-∞(ytendstominusinfinity)

Inotherwordsyissaidhaveaninfinitelimitasx→a.Andwe write

$$\lim_{x \to a} y = -\infty$$

Example

If

$y = -\frac{1}{x^2},$

Then

$$\lim_{x \to 0} y = -\infty$$

Sincex→0, $x^2 \to 0 \, and \, x^2$ispositive,

$-\frac{1}{x^2}$ becomesveryverylargeandisnegative.Thereforetheresult.

LIMITAT INFINITY

As x becomes very very large or in other words the value of x is greater than a verylargepositivenumber,i.e.x→∞,ifvalueofyisclosetoafinitevalue'a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x \to \infty} y = a$$

Example

Let$y = \frac{1}{x}$

Asx→∞ , $\frac{1}{x}$ becomesveryverysmallandapproachesthevalue 0.Therefore wewrite

$$\lim_{x \to \infty} y = 0$$

similarly

As x becomes very very large with a negative sign or in other words the value ofxislessthanaverylargenegativenumber,i.e.x→-∞,ifvalueofyisclose to a finite value' a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x\to-\infty} y=a$$

Example

Let$y=\frac{1}{x}$

Asx→∞ , $\frac{1}{x}$ becomesveryverysmallandapproachesthevalue 0.Therefore wewrite

$$\lim_{x\to-\infty} y=0$$

ALGEBRAOFLIMITS

1. Limitofsumoftwofunctionsissumoftheirindividuallimits Let

$\lim_{x\to a} f(x)=m\ and\ let \lim_{x\to a} g(x)=n$, then

$$\lim_{x\to a}(f(x)+g(x))=m+n$$

2. Limitofproductoftwofunctionsisproductofthetheirindividuallimits Let

$\lim_{x\to a} f(x)=m\ and\ let \lim_{x\to a} g(x)=n$, then

$$\lim_{x\to a}(f(x)\times g(x))=m\times n$$

3. Limitofquotientoftwofunctionsisquotientofthetheirindividual limits

Let$\lim_{x\to a} f(x)=mandlet \lim_{x\to a} g(x)=n\neq 0$,then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{m}{n}$$

SOMESTANDARDLIMITS

1. $\lim_{x\to a} P(x) = P(a)$ where P(x) is polynomial in x

Example

$$\lim_{x\to 1}(2x^2+3x+1)=2\times 1^2+3\times 1+1=6$$

2. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ wherenisarationalnumber

Example

$$\lim_{x\to a}\frac{x^2-a^2}{x-a}=2a^{2-1}=2a$$

3. $\lim_{n\to\infty}(1+\frac{1}{n})^n=e,$

$\lim_{n\to\infty}(1+\frac{k}{n})^n=e^k,$

4. $\lim_{n\to 0}(1+n)^{\frac{1}{n}}=e$

$\lim_{n\to 0}(1+n)^{\frac{k}{n}}=e^k$

5. $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\ln a$

Example

$.\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)=\ln 2$

6. $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e$

Example

$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\ln e=1$

SOMESTANDARDTRIGONOMETRICLIMITS

1$\lim_{x\to 0}\sin x=0$

2. $\lim_{x\to 0}\cos x=1$

3. $\lim_{x\to 0}\tan x=0$

4. $\lim_{x\to 0}\frac{\sin x}{x}=1$, here $x\to 0 through radian values$

5. $\lim_{x\to 0}\frac{\sin x^0}{x}=\frac{\pi}{180}$

Example

$$\lim_{x\to 0}\frac{\sin mx}{\sin nx}=\frac{m}{n}$$

Since

$$\lim_{x\to 0}\frac{\sin mx}{\sin nx}=\lim_{x\to 0}\frac{\sin mx}{mx}\times\frac{nx}{\sin nx}\times\frac{m}{n}$$

$$=1\times 1\times\frac{m}{n}=\frac{m}{n}$$

$$\lim_{x\to\infty}\frac{2x^2+3x+5}{3x^2+2x+1}=\lim_{x\to\infty}\frac{2+\frac{3}{x}+\frac{5}{x^2}}{3+\frac{2}{x}+\frac{1}{x^2}}=\frac{2}{3}$$

Example

$$\lim_{x\to\infty}\frac{1+2+3\ldots\ldots\ldots+n}{n^2}$$

$$=\lim_{x\to\infty}\frac{n(n+1)}{2\times n^2}=\lim_{x\to\infty}\frac{(n^2+n)}{2\times n^2}=\lim_{x\to\infty}\frac{(1+\frac{1}{n})}{2}=\frac{1}{2}$$

Example

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \frac{\sin^2 x}{2\frac{x^2}{4}} = \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\times\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)$$

$$=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

ExistenceofLimits

Whenwesayxtendsto'a'orwritex→aitcanhappenintwodifferentways

Xcanapproach'a'throughvaluesgreaterthan'a'i.efromrightsideof'a'on the Number Line

Or

Xcanapproach'a'throughvaluessmallerthan'a'i.efromleftsideof'a'onthe Number Line

ThefirstcaseiscalledtheRightHandLimitandthelatercaseis calledtheLeft Hand Limit.

We,thereforeconcludethatLimitwillexistifftherightHandLimitandtheLeft Hand Limit both exist and are EQUAL

ConsidertheGreatestIntegerFunction

$$y=f(x)=[x]$$

Considerthelimitof thisfunctionas$x\to 1$

Therighthandlimitofthisfunction

$$\lim_{x\to 1+}[x]=1$$

Sinceifthevalueofxisgreaterthan1for example1+h,h$>$ 0,thenthegreatest integer less than equal to 1+h is 1

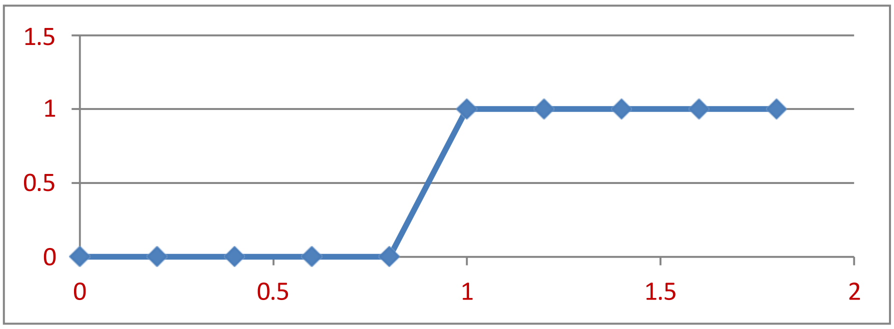
Thelefthandlimitofthis function

$$\lim_{x\to 1-}[x]=0$$

Sinceifthevalueofxislessthan1forexample1-h,h> 0,thenthe greatest integer less than equal to 1-h is 0

Inthis casetherighthandlimitandthelefthandlimitarenotequal And

therefore the limit of this function as $x\rightarrow$ 1does not exist




Forthatmatterthisfunctiondoesnotallowlimitas$x\rightarrow n$

Sincetherighthandlimitwillbealwaysnandthelefthandlimitwillben-1.

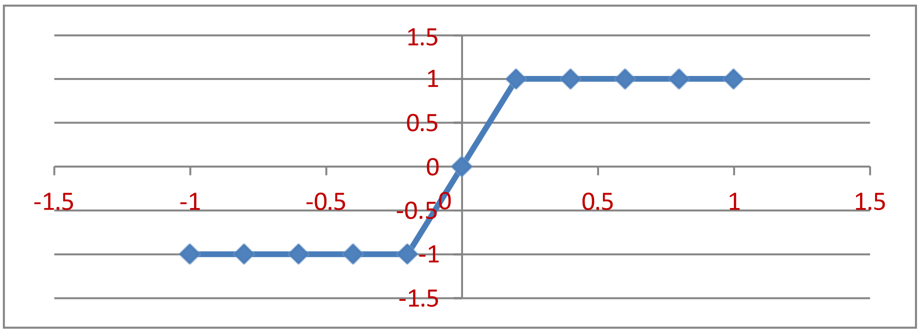
ConsidertheSignumFunction

$$y=f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

Considerthelimitofthisfunctionas$x\rightarrow 0$

The right hand limit of this function is 1 and the left hand limit of this function is-1asevidentfromthedefinitionofthefunctionandconceptofrightandleft hand limits

Thereforethisfunctiondoesnothavealimitas$x\rightarrow 0$



# Differentiation

Afunctionf(x)issaidtobedifferentiableatapointx=ciff

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \quad exists$$

Ingeneral,afunctionisdifferentiableiff

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad exists$$

Oncethislimitexists,itiscalledthedifferentialcoefficientoff(x) orthe derivative of the function f(x) at x=c

Or

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Where $f'(c)$ and $f'(x)$ are the differential coefficient or the derivative ofthe function, the first being defined at x=c

Examples

Considerthefunction

$$y=f(x)=kortheconstantfunction$$

Inthiscasethedifferentialcoefficient$f'(x)$isgiven by

$$f'(x)=\lim_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0} \frac{k-k}{\delta x} = 0$$

Thereforetheconstantfunctionisdifferentiableeverywhereandthe derivative is zero

Considerthefunction

$$y=f(x)=x^2$$

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{(x+\delta x)^2-x^2}{(x+\delta x)-x}$$

$$=2x$$

Considerthefunction

$$y=f(x)=\sin x$$

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{sin(x+\delta x)\quad -sinx}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{2cos\left(\frac{x\pm\delta x\pm x}{2}\right)\quad\times sin\left(\frac{x\pm\delta x-x}{2}\right)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{cos\left(\frac{x\pm\delta x\pm x}{2}\right)\quad\times sin\left(\frac{x\pm\delta x-x}{2}\right)}{\frac{\delta x}{2}}$$

$$=\frac{cos\left(\frac{x\pm\delta x\pm x}{2}\right)\quad\times sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$=\lim_{\delta x\to 0}cos\left(x+\frac{\delta x}{2}\right)\times 1$$

$$=\cos x$$

Therefore

$$y=f(x)=sinx$$

$$\frac{dy}{dx} = \cos x$$

## *Algebraofderivatives*

Considertwodifferentiablefunctionsu(x)andv(x) Let

$$y = u + v$$

Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Let

$$y = u \times v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let

$$y = \frac{u}{v}, \qquad v \neq 0$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

1

$$y = \sin x + x^3$$

$$\frac{dy}{dx} = \cos x + 2x$$

2

$$y = x^2 \cos x$$

$$\frac{dy}{dx}=x^2(-sinx)+cosx(2x)$$

$$=-x^2sinx+2xcosx$$

3

$$y=\frac{sinx}{cosx}$$

$$\frac{dy}{dx}=\frac{cosxcosx-sinx(-sinx)}{(cosx)^2}$$

$$\frac{dy}{dx}=\frac{(cosx)^2+(sinx)^2}{}(cosx)^2$$

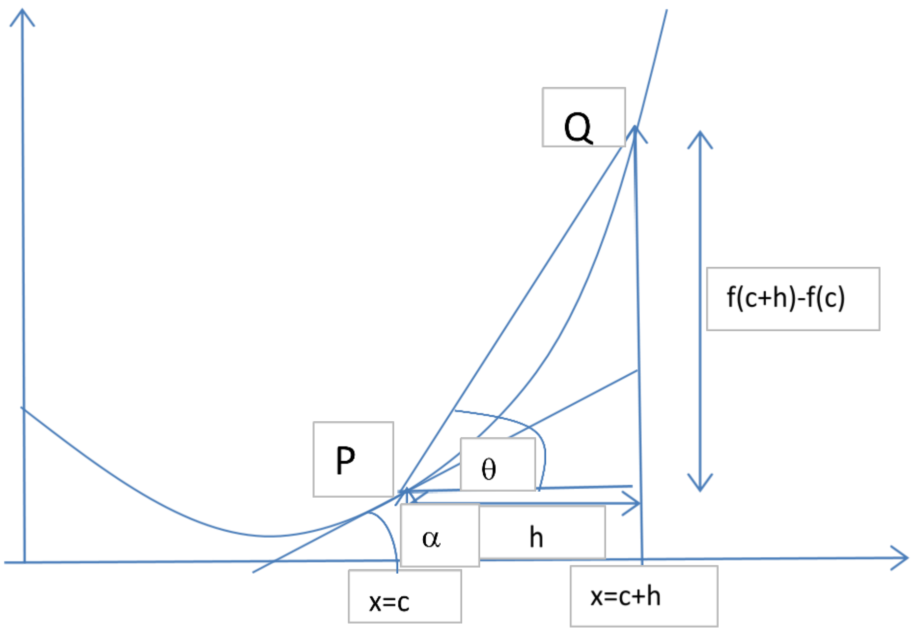
$$\frac{dy}{dx}=\frac{(cosx)^2+(sinx)^2}{}(cosx)^2$$

$$\frac{dy}{dx}=\frac{1}{(cosx)^2}=(secx)^2$$

**Geometricalmeaningof** $f'(c)$

Considerthegraphofa function

$$y=f(x)$$



$$\underline{f(c+h)-f(c)\ h}$$

Representstheratioofheighttobaseoftheanglethelinejoiningthepoint P(c,f(c )) and Q (c+h,f(c+h))

i.e

$$\frac{f(c+h)-f(c)\ h}{} = tan\theta$$

Where$\theta$ istheanglethelinejoiningthepointPandQmakeswith thepositive direction of x axis.

Inthelimitingcase as h→ 0i.eas Q→ $P$ thelinePQ becomes thetangent line andthetheangle $\theta\ becomes\ the\ angle\ \alpha$ whichthetangentlinemakeswith the positive direction of x axis

i.e

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)=tan\alpha=m(theslopeofthetangent)$$

ApplicationtoGeometry

Tofindtheequationofthetangentlinetothecurvey=f(x)atx=$x_0$

The equation oflinepassing through thepoint($x_0$,$f(x_0)$)isgive by

$$y-f(x_0)=m(x-x_0)$$

Where‘m’istheslopeofthetangentline.

As,wehaveseen

$$m=f'(x_0)$$

Theequationistherefore

$$y-f(x_0)=f'(x_0)(x-x_0)$$

Intheaboveexampleifwetake

$f(x)=x^2 andthepointx_0=1$

Theequationtothetangentatthepointisgiven by

$$y-f(x_0)=f'(x_0)(x-x_0)$$

Or

$$y-1^2=2\times 1(x-1)$$

where

$$f(x_0)=x^2=1^2 and f'(x_0)=2\times x_0=2\times 1$$

i.e

theequationis

$$y-1=2(x-1)$$

## *Derivativeasratemeasurer*

Rememberthedefinition

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$$

Thequantity

$$\frac{f(c+h)-f(c)}{h}$$

$measurestherateofchangeinf(c)wthrespecttochangehin \quad 'c'$

Considerthelinearmotionofaparticlegiven as

$$s=f(t)$$

Where‘s’denotesthedistancetraversedand‘t’denotesthetimetaken

Theratio

$$\frac{s}{t}$$

Denotesthe**averagevelocity**oftheparticle

Tocalculatethelocalvelocityorinstantaneousvelocityatapointoftime t=$t_0$we proceed in the following way

Consideraninfinitesimaldistance’ $\delta s'$traversedfromtimet=$t_0$intime’$\delta t'$

Theratio

$$\frac{\delta s}{\delta t}$$

Stillrepresentsaaveragevalueofthevelocity

Theinstantaneousvelocityatt=$t_0$canbecalculatedbyconsideringthe following limit

$$\lim_{\delta t\to 0}\frac{\delta s}{\delta t}$$

or

$$v= \frac{ds}{dt}$$

Where‘v’representstheinstantaneous***velocity***whichisdefinedasrateof change of displacement

Similarly,wecanwritethemathematicalexpressionfor***acceleration***

As

$$a= \frac{dv}{dt}$$

Ortherateofchangeof velocity

Example

Ifthemotionofaparticleisgiven by

$$s=f(t)=2t+5$$

Whichislinearinnature,wecancalculatevelocityatt=3

$$v(t=3)= \frac{ds}{dt} =2$$

Itisclearthatthevelocityisindependentoftime ‘t’.

i.e

theabovemotionhasconstantoruniformvelocity.

And,therefore,theacceleration

$$a= \frac{dv}{dt}=0$$

Orthemotiondoesnotproduceanyacceleration. Consider

another motion of a particle given as

$$s=f(t)=2t^2+3$$

Herethevelocityatt=3canbecalculatedas

$$v(t=3)= \frac{ds}{dt}=4t=4\times3=12$$

Andtheacceleration

$$a=\frac{dv}{dt}=4$$

Thereforewecansaythatthemotionissaidtohaveconstantor uniform acceleration

## *Derivativesofimplicitfunction*

Considertheequationofacircle

$$x^2+y^2=r^2$$

Thisisanimplicitfunction

Letsdifferentiatethisequationwithrespectxthroughout,we get

$$2x+2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}= \frac{-x}{y}$$

## *Derivativeofparametricfunction*

Theequationofacirclecanalsobewritten as

$$x=rcost$$

$$y=rsint$$

Thisiscalledparametricfunctionhavingparameter ‘t’

Inthiscase

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{rcost}{-rsint}=\frac{x}{-y}=\frac{-x}{y}$$

## *Derivativeoffunctionwithrespecttoanotherfunction*

Considerthefunctions

$$y=f(x)$$

$$z=g(x)$$

$$\frac{dy}{dx}=\frac{f'(x)}{g'(x)}$$

Example

Let

$$y=sin(x)$$

$$z=x^3$$

$$\frac{dy}{dx}=\frac{f'(x)}{g'(x)}=\frac{cosx}{3x^2}$$

Derivativeofcompositefunction

Considerthefunction

$$y=f(u)\ whereu=g(x)$$

Thenyiscalledacompositefunction

Inthiscase

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

ThisiscalledChainRule.Thiscanbeextendedtoanynumberoffunctions. Example

1. Let

$$y = sinx^2$$

Thiscanbewritten as

$$y = sinu$$

And

$$u = x^2$$

Applyingchainrule, wehave

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = cosu \times 2x = 2xcosx^2$$

2. Let

$$y = tane^{x^2}$$

Thiscanbewritten as

$$y = tanu$$

And

$$u = e^v$$

$$v = x^2$$

Applyingchainrule, wehave

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}=sec^2u\times e^v\times 2x=sec^2e^{x^2}\times e^{x^2}\times 2x$$

*Derivativesofinversefunction*

$$since \quad \frac{\delta x}{\delta y}=\frac{1}{\frac{\delta y}{\delta x}}$$

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$$

As $\delta x\to 0, \delta y also \to 0$

Whichfollowsfromthefactthat

$y=f(x)beingadifferentiablefunctionisacontinuousfunction$

Andtheconditionofcontinuityguaranteestheabovefact.

*Derivativeofinversetrigonometricfunction*

Let

$$y=sin^{-1}x$$

Where $y\epsilon(-\frac{\pi}{2},\frac{\pi}{2})$

Thiscanbewritten as

$$x=siny$$

$$\frac{dx}{dy}=cosy$$

Or

$$\frac{dy}{dx}=\frac{1}{cosy}=\frac{1}{\pm\sqrt{(1-sin^2y)}}=\frac{1}{\pm\sqrt{(1-x^2)}}=\frac{1}{\sqrt{(1-x^2)}}$$

Sincecosyispositiveinthedomain

Let

$$y=cos^{-1}x$$

Where $y\epsilon(0,\pi)$

Thiscanbewritten as

$$x=cosy$$

$$\frac{dx}{dy}=-siny$$

Or

$$\frac{dy}{dx}=\frac{-1}{siny}=\frac{-1}{\pm\sqrt{(1-cos^2y)}}=\frac{-1}{\pm\sqrt{(1-x^2)}}=\frac{-1}{\sqrt{(1-x^2)}}$$

Sincesinyispositiveinthedomain

Let

$$y=sec^{-1}x$$

Where $y\epsilon(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$

Thiscanbewritten as

$$x=secy$$

$$\frac{dx}{dy}=secy\times tany$$

Or

$$\frac{dy}{dx}=\frac{1}{secy\times tany}=\frac{1}{x\sqrt{(sec^2y-1)}}=\frac{1}{x(\pm\sqrt{(x^2-1)})}=\frac{1}{|x|\sqrt{(1-x^2)}}$$

Since$secy\times tany$ispositiveinthedomain

Let

$$y=cosec^{-1}x$$

Where $y\epsilon(-\frac{\pi}{2},0)\cup(0,\ \frac{\pi}{2})$

Thiscanbewrittenas

$$x=cosecy$$

$$\frac{dx}{dy}=-cosecy\times coty$$

Or

$$\frac{dy}{dx}=\frac{-1}{cosecy\times coty}=\frac{-1}{x\sqrt{(cosec^2y-1)}}=\frac{-1}{x(\pm\sqrt{(x^2-1)})}=\frac{-1}{|x|\sqrt{(1-x^2)}}$$

Since$cosecy\times coty$is positiveinthedomain

Let

$$y=tan^{-1}x$$

Thiscanbewritten as

$$x=tany$$

$$\frac{dx}{dy}=sec^2y$$

Or

$$\frac{dy}{dx}=\frac{1}{sec^2y}=\frac{1}{1+tan^2y}=\frac{1}{1+x^2}$$

Let

$$y=cot^{-1}x$$

Thiscanbewritten as

$$x=coty$$

$$\frac{dx}{dy}=-cosec^2y$$

Or

$$\frac{dy}{dx}=\frac{-1}{cosec^2 y}=\frac{-1}{1+cot^2 y}=\frac{-1}{1+x^2}$$

## *Higherorderderivatives*

Let

$$y=f(x)$$

Isdifferentiableandalso

$$\frac{dy}{dx}=f'(x)$$

Isdifferentiable.Thenwedefine

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=f''(x)$$

$$=$$

$$=\lim_{\delta x\to 0}\frac{f'(x+\delta x)-f'(x)}{\delta x}$$

Thisisthe2nd. Orderderivativeofthefunction

Similarlywecandefinehigherorderderivativesofthefunction Example

Let

$$y=f(x)=x^3+x^2+x+1$$

$$\frac{dy}{dx}=f'(x)=3x^2+2x+1$$

$$\frac{d^2y}{dx^2}=f''(x)=6x+2$$

ConsidertheFunction

$$y=f(x)=Acosx+Bsinx$$

Here

$$\frac{dy}{dx}=f'(x)=-Asinx+Bcosx$$

$$\frac{d^2y}{dx^2}=f''(x)=-Acosx-Bsinx=-y$$

i.einthiscase

$$\frac{d^2y}{dx^2}+y=0$$

## *MonotonicFunction*

## *Increasing function*

Considera function

$$y=f(x)$$

If$x_2>x_1 implies f(x_2)>f(x_1)$

Thenthefunctionisincreasing

Example

$$y=f(x)=x+1$$

$$f(2)=2+1=3$$

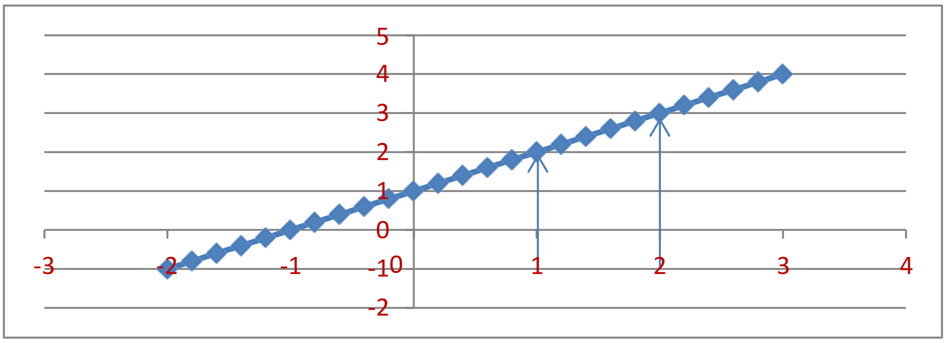
$$f(1)=1+1=2$$

Or

$$f(2)>f(1)$$

Thereforethefunctionisincreasing Graph

of the function




*Decreasingfunction*

Considerafunction

$$y=f(x)$$

If$x_2>x_1 implies f(x_2)<f(x_1)$

Thenthefunctionis decreasing

Considerthefunction

$$y=f(x)=\frac{1}{x}$$

$$f(2)=\frac{1}{2}$$

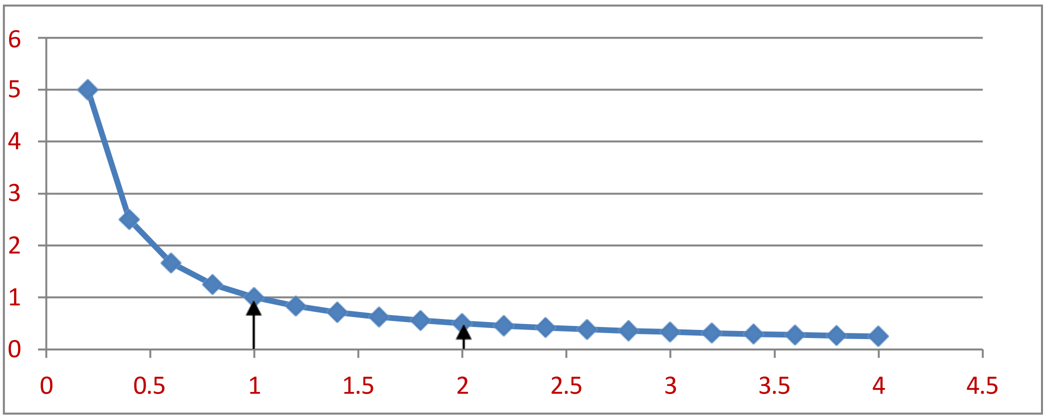
$$f(1)=\frac{1}{1}=1$$

Or

$$f(2)<f(1)$$

Thereforethefunctionisdecreasing Graph

of the function




Afunctioneitherincreasingordecreasingiscalledmonotonic.

Derivative of Increasing Function

Iff(x)isincreasing,then

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}>0$$

i.e

forincreasingfunctionthederivativeisalwayspositive

DerivativeofDecreasingFunction

Iff(x)isdecreasing,then

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}<0$$

i.e

fordecreasingfunctionthederivativeisalwaysnegative

example

let

$$y=f(x)=x+1$$

$$\frac{dy}{dx}=f'(x)=1>0$$

Thereforethefunctionisincreasing

Let

$$y=f(x)=\frac{1}{x}$$

$$\frac{dy}{dx}=f'(x)=\frac{-1}{x^2}<0$$

Thereforethefunctionisdecreasing

Let

$$y=f(x)=x^2$$

$$\frac{dy}{dx}=f'(x)=2x$$

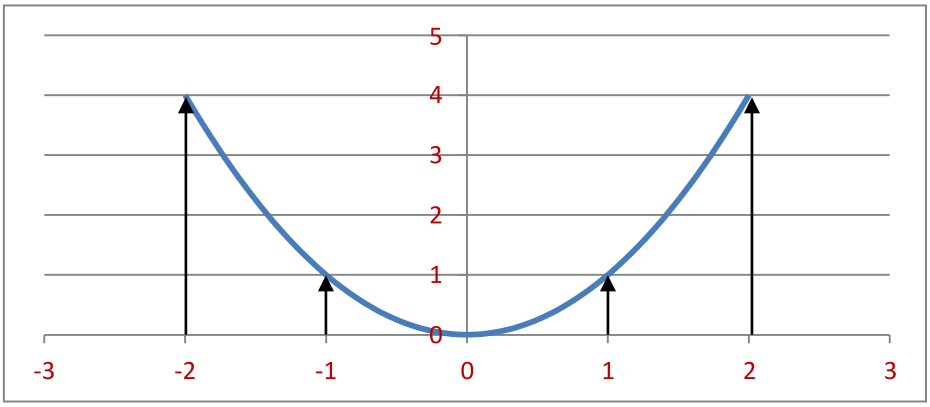
$$>0 for x>0$$

$$<0 for x<0$$

Thereforethefunctionisincreasingfor$x>0$anddecreasingfor$x<0$

Graphofthefunction

$$y=f(x)=x^2$$



MAXIMAANDMINIMAOFA FUNCTION

Considera function

$$y=f(x)$$

Considerthepoint x=c

Ifatthispoint

$$f(c)>f(c+h), where |h|<\delta$$

Thenf(c)iscalledlocalmaximumorsimplyamaximumofthe function

Ifatthispoint

$$f(c)<f(c+h), where |h|<\delta$$

Thenf(c)iscalledalocalminimumorsimplya minimum

A function can have several local maximum values and several local minimum valuesinitsdomainanditispossiblethatalocalminimumcanbelargerthana local maximum.

Iff(c)isalocalmaximumthenthegraphofthefunctioninthedomain

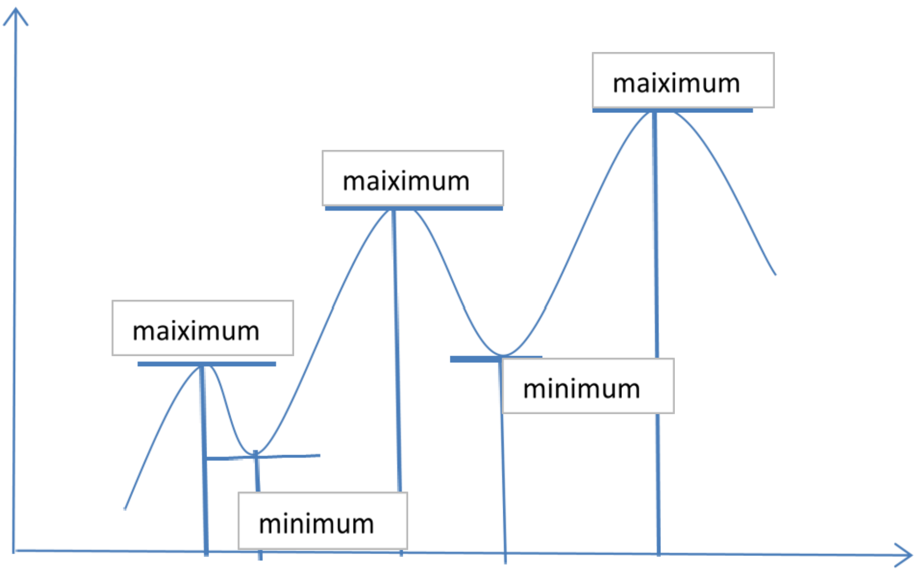
$$(c-\delta, c+\delta)$$

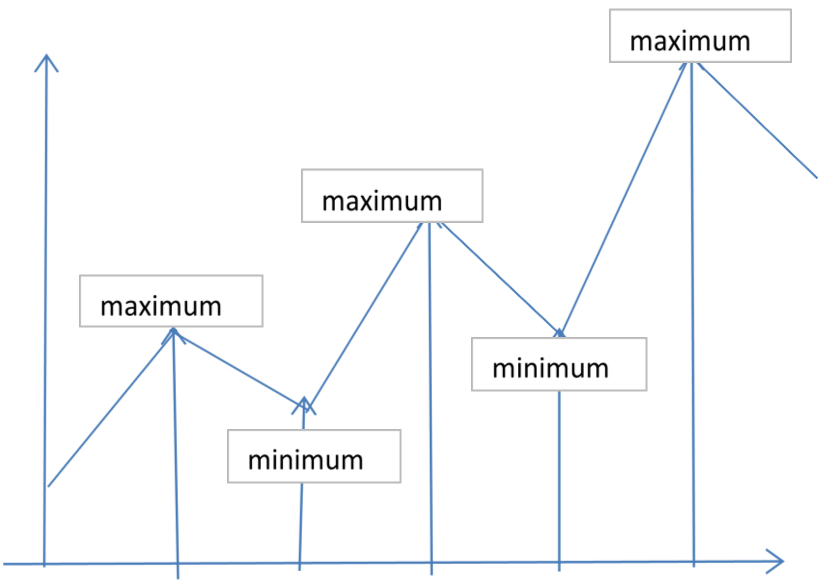
Willbeconcavedownwards

Iff(c)isalocalminimumthenthegraphofthefunctioninthedomain

$$(c-\delta, c+\delta)$$

Willbeconcaveupwards

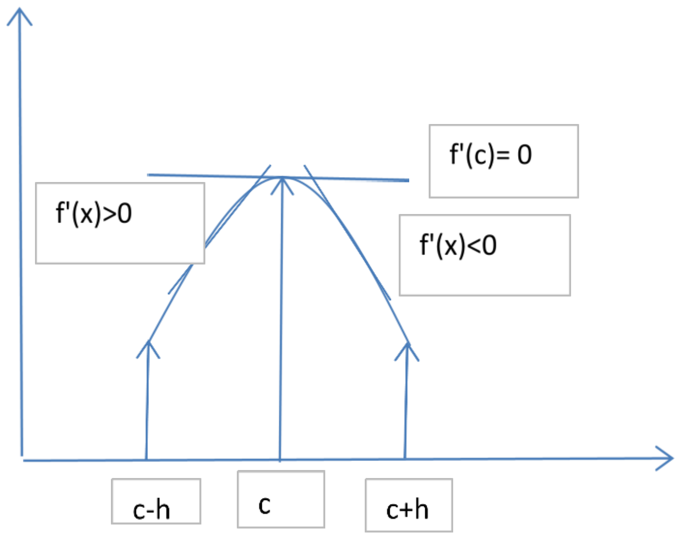


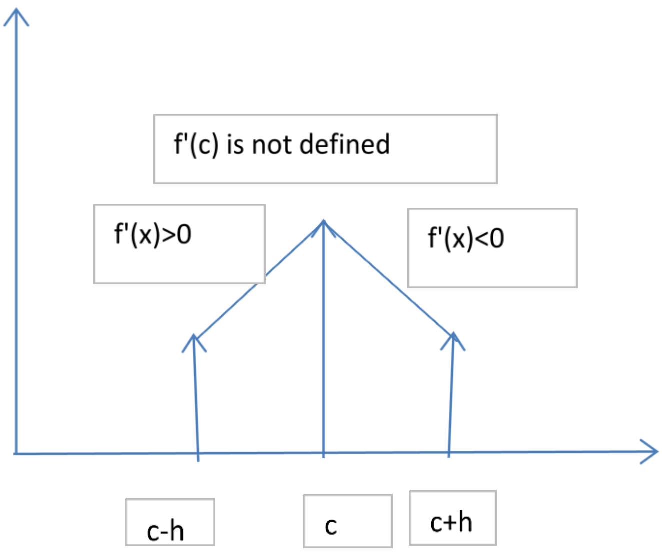




*MaximumCase*

Inotherwordsatapointoflocalmaximumthefunctionisincreasingonthe left of the point and decreasing on the right of the point

Thereforethederivativeofthefunctionchangessignfrompositivetonegative as it passes through x=c



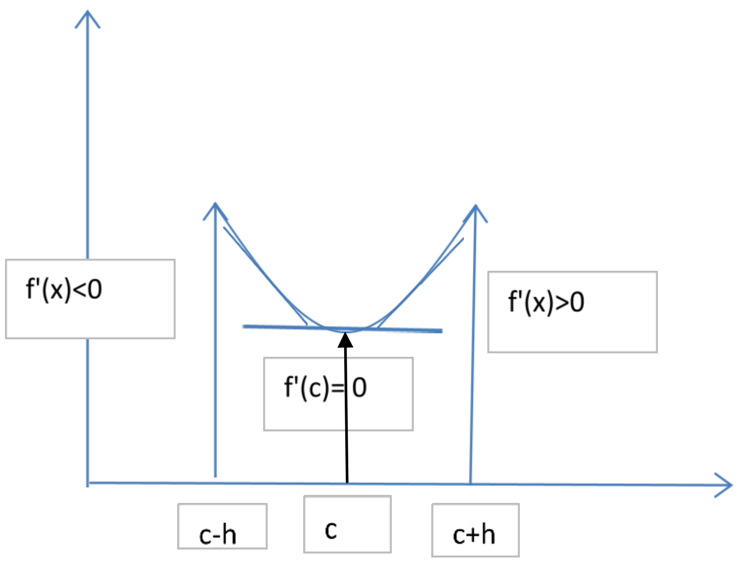



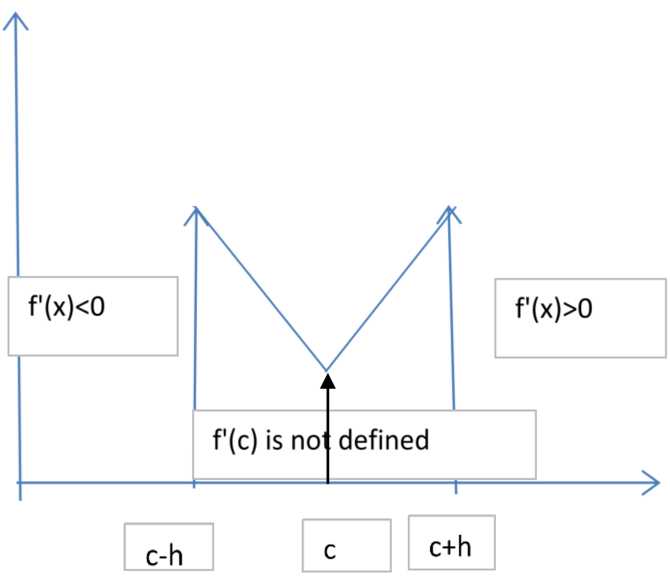

Therefore we conclude that the derivative of the function is a decreasing functionandassuchitsderivativei.ethesecondorderderivativeisnegative

*MinimumCase*

Atapointoflocalminimumthefunctionisdecreasingontheleftof thepoint and increasing on the right of the point

Thereforethederivativeofthefunctionchangessignfromnegativetopositive as it passes through x=c



Therefore we conclude that the derivative of the function is a increasing functionandassuchitsderivativei.ethesecondorderderivativeispositive

Ineithermaximumorminimumcasethe1st.derivativeofthefunctioniszero or is not defined at the point of maximum or minimum

Thepointx=cwherethederivativevanishesordoesnotexistatalliscalleda critical point or turning point or stationary point.

Afunctioncanhaveneitheramaximumnoraminimumvalue Example

Considerthefunction

$$y=f(x)=x^3$$

Here

$$\frac{dy}{dx}=3x^2$$

Thisvanishesat x=0

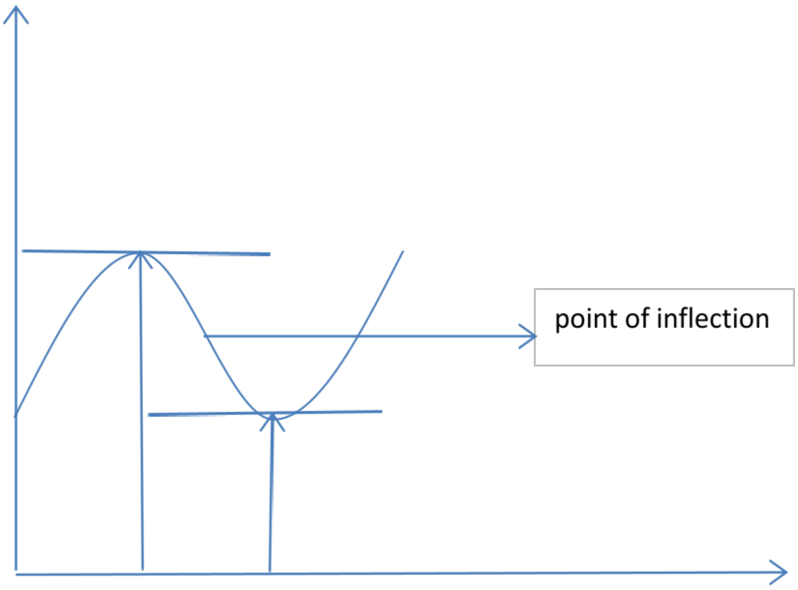
And

$$\frac{d^2y}{dx^2}=6x$$

Whichalso vanishes

Thereforewemayconcludethatthefunctiondoesnothavemaximumneither minimum value

Pointof inflexion




If a curve is changing its nature from concave downwards to concave upwards asshowninthefigureorviceversa,thenatthepointwherethischangeoccurs is called the point of inflexion. In other words on one side of the point of inflexion the curve is concave downward and on the other side the curve is concave upward or vice versa

Intheabovefigure,

Ontheleftsideofpointofinflexionamaximumvalueoccursandtotheright side of point of inflexion a minimum value occurs.

Inotherwords,rememberingtheconditionofmaximumandminimum,wecan say,

The$2^{nd}$orderderivativechangesitssignfromnegativetopositiveasinthe case given in the figure or vice versa.

Inotherwordsthepointofinflexionisthepointofeithermaximumor minimum of the $1^{st}$ derivative of the function

Hence at the point of inflexion the $2^{nd}$. order derivative vanishes or is not definedandthe$2^{nd}$. orderderivativechangesitssignasitpassesthroughthe point of inflexion i.eatthepointofinflexion

1. $\frac{d^2y}{dx^2}=f''(x)=0$orisnot defined
2. The2nd.orderderivativechangessignasitpassesthroughthepoint

Example

Considerthefunctionwediscussedearleier

$$y=f(x)=x^3$$

Here

$$\frac{dy}{dx}=3x^2$$

Thisvanishesat x=0

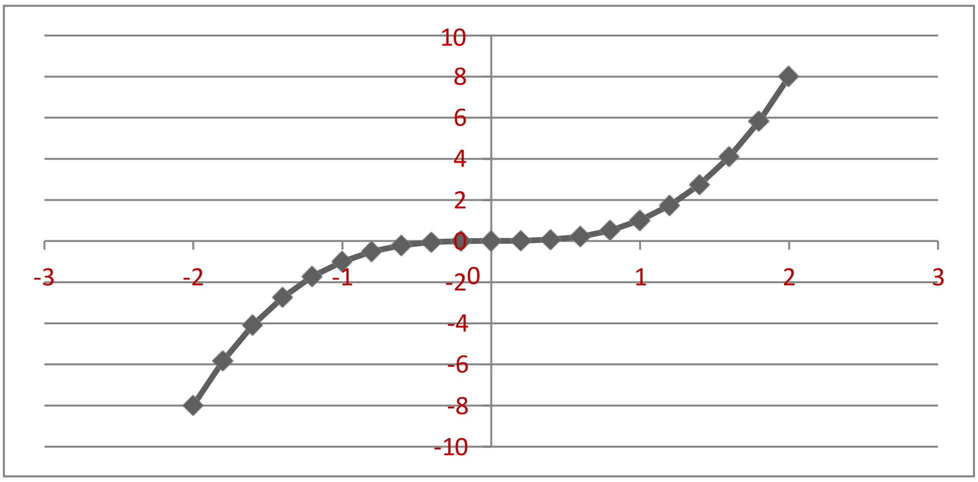
And

$$\frac{d^2y}{dx^2}=6x$$

Whichalsovanishesatx=0

But

$$\frac{d^3y}{dx^3}=6\neq 0$$




Thereforeweconcludethat

X=0isapointofinflexionforthecurve

Workingproceduretofindthemaximaandminima

1. Givenanyfunction,equatethefirstderivativetozerotofindtheturning points or critical points

2. Testthesignofthesecondderivativeatthesepoints.Ifthesignisnegativeit is appoint of maximum value. If the sign is positive it is a point of minimum value.

3. thencalculatethemaximumvalue/minimumvalueofthefunctionbytaking the value of x as the point

Example

Ifthesumoftwonumbersis10,findthenumberswhentheirproductis maximum

Solution

Letthenumbersbexand10-x

Let

$$y=f(x)=x(10-x)$$

$$=10x-x^2$$

$$\frac{dy}{dx}=10-2x=0$$

$$x=5$$

$$\frac{d^2y}{dx^2}=-2<0$$

Thereforethefunctionwhichistheproductofthenumbersmaximumifthe numbers are equal i.e 5 and 5.

EXAMPLE

Investigatetheextremevaluesofthefunction

$$f(x)=x^4-2x^2+3$$

Thecriticalpointsarerootsoftheequation